AN INDIRECT SUFFICIENCY PROOF FOR THE PROBLEM OF LAGRANGE WITH DIFFERENTIAL INEQUALITIES AS ADDED SIDE CONDITIONS

BY

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1. Introduction. The problem to be considered here consists in finding in a class of arcs C: $y^i(x)$ $(i=1, \cdots, n;\ x^1 \leqq x \leqq x^2)$ joining two fixed points and satisfying a set of differential inequalities and equations of the form

$$\phi^\beta(x, y, \dot{y}) \geqq 0, \qquad \psi^\rho(x, y, \dot{y}) = 0$$

that one which minimizes the integral

$$I(C) = \int_{x^1}^{x^2} f(x, y, \dot{y}) dx.$$

Valentine [11](1) has given a brief history of this problem and has derived certain necessary conditions by introducing auxiliary functions $z^\beta(x)$ such that $\phi^\beta(x, y, \dot{y}) = \dot{z}^{\beta^2}$. His sufficiency theorems depended on assumptions of normality and a field theory, and also required all except one of the differential inequalities to be satisfied in the strict sense along the minimizing arc C_0.

Using methods developed by McShane [9] and Hestenes [3; 4; 5; 6] we shall give an indirect proof of a sufficiency theorem. Instead of demanding that all but one of the functions ϕ^β is positive along C_0 we shall impose the more general restriction (2.5) to be described in the next section.

2. Statement of problem and main theorem. Let $\mathcal{R}$ be a region in $(2n+1)$-dimensional space of points $(x, y, p) = (x, y^1, \cdots, y^n, p^1, \cdots, p^n)$. By an admissible arc C will be meant a set of functions $y^i(x)$ $(i=1, \cdots, n;\ x^1 \leqq x \leqq x^2)$ which are absolutely continuous and have integrable square derivatives $\dot{y}^i(x)$ such that the point $[x, y(x), \dot{y}(x)]$ is in $\mathcal{R}$ for almost all x on x^1x^2. It will be assumed that the functions $f(x, y, p)$, $\phi^\beta(x, y, p)$, $\psi^\rho(x, y, p)$ are of class C'' on $\mathcal{R}$ $(\beta=1, \cdots, m;\ \rho=m+1, \cdots, m+t<n)$. The subset on $\mathcal{R}$ on which $\phi^\beta(x, y, p) \geqq 0$, $\psi^\rho(x, y, p) = 0$ will be denoted by $\mathcal{D}$. We shall say that C lies in $\mathcal{D}$ if the point $[x, y(x), \dot{y}(x)]$ lies in $\mathcal{D}$ for almost all x on x^1x^2.

We shall be concerned with a particular admissible arc C_0: $y^i = y_0^i(x)$ of class C' which lies in $\mathcal{D}$, satisfies the end conditions

$$(2.1) \qquad y^i(x^s) = y^{is} \qquad (i = 1, \cdots, n;\ x^1 \leqq x \leqq x^2;\ s = 1, 2)$$

and along which the matrix

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(1) The numbers in brackets refer to the bibliography at the end of the paper.

$$\left\| \begin{matrix} \phi^{\beta}_{p^i} \\ \psi^{\rho}_{p^i} \end{matrix} \right\| \tag{2.2}$$

has rank $m+t$. Suppose that the interval x^1x^2 can be divided into a finite number of open intervals A_τ: $x_\tau < x < x_{\tau+1}$ ($\tau = 0, 1, \cdots, T$), where $x_0 = x^1$, $x_{T+1} = x^2$, in such a manner that each of the functions $\phi^\beta[x, y_0(x), \dot{y}_0(x)]$ either vanishes identically on A_τ or is positive everywhere on A_τ. Let $\Gamma(x)$ be the set of indices β such that $\phi^\beta[x, y_0(x), \dot{y}_0(x)] = 0$. Then $\Gamma(x)$ is independent of x when x is in A_τ and may be denoted by Γ_τ. Let $A(\beta)$ be the closure of the sum of the intervals A_τ on which $\phi^\beta[x, y_0(x), \dot{y}_0(x)] = 0$.

Consider now a set of continuous functions $\lambda^0 \geqq 0$, $\lambda^\beta(x)$, $\lambda^\rho(x)$ such that if we define

$$F(x, y, p, \lambda) = \lambda^0 f(x, y, p) + \lambda^\beta \phi^\beta(x, y, p) + \lambda^\rho \psi^\rho(x, y, p),$$

then the equations

$$F_{p^i} = \int_{x^1}^{x} F_{y^i} dx + c^i, \tag{2.3}$$

$$\lambda^\beta(x)\phi^\beta(x, y, \dot{y}) = 0 \qquad (\beta \text{ not summed}) \tag{2.4}$$

hold along C_0 with the multipliers $\lambda(x)$ for some set of constants c^i. Let $\Delta(x)$ be the set of indices β such that $\lambda^\beta(x) \neq 0$ and let $B(\beta)$ be the set of points x for which $\lambda^\beta(x) \neq 0$. It is evident from (2.4) that $\overline{B}(\beta)$, the closure of $B(\beta)$, is contained in $A(\beta)$ and that $\Delta(x)$ is contained in $\Gamma(x)$. In fact, $\Delta(x)$ is contained in Γ_τ for all x in $\overline{A}_\tau$. We shall assume that

$$\Gamma(x) - \Delta(x) \text{ contains at most one index.} \tag{2.5}$$

We shall make a further restriction on our choice of multipliers. Let $E_F(x, y, p, q, \lambda)$ be the Weierstrass E-function

$$E_F = F(x, y, q, \lambda) - F(x, y, p, \lambda) - (q^i - p^i)F_{p^i}(x, y, p, \lambda),$$

and define $\bar{\phi}^\beta(x, y, p)$ as

$$\bar{\phi}^\beta(x, y, p) = \phi^\beta(x, y, p)/[1 + \phi^{\beta^2}(x, y, p)]^{1/2}.$$

It will be assumed that there is a neighborhood $\mathfrak{D}_1$ of C_0 relative to the set $\mathfrak{D}$ and a constant b such that $0 < b < 1$ and the inequality

$$E_F(x, y, p, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, q) \geqq bE_L(p, q) - \lambda^\beta(x)\bar{\phi}^\beta(x, y, q) \tag{2.6}$$

holds whenever (x, y, q) is in $\mathfrak{D}_1$, (x, y, q) is in $\mathfrak{D}$, and $\phi^\beta(x, y, p) = 0$ in case β is in $\Delta(x)$. Here $L(p)$ is the integrand $(1 + p^i p^i)^{1/2}$ of the length integral and $E_L(p, q)$ is the E-function

$$E_L(p, q) = L(q) - (1 + p^i q^i)/L(p) \tag{2.7}$$

for this integrand. We shall call the set of functions $\lambda^0 \geqq 0$, $\lambda^\beta(x)$, $\lambda^\rho(x)$ an *admissible set of multipliers* when the functions are continuous on x^1x^2 and the conditions (2.3), (2.4), (2.5), and (2.6) are satisfied under the conditions described.

Let us now consider a set of functions $\eta^i(x)$ which are absolutely continuous and have integrable square derivatives $\dot{\eta}^i(x)$ on x^1x^2. If such a set of functions satisfies with the curve C_0 and an admissible set of multipliers the end conditions

$$\eta^i(x^s) = 0, \tag{2.8}$$

the differential equations

$$\phi^\beta_{y^i}\eta^i + \phi^\beta_{p^i}\dot{\eta}^i = 0 \tag{2.9}$$

for almost all x in $B(\beta)$, the differential inequalities

$$\phi^\beta_{y^i}\eta^i + \phi^\beta_{p^i}\dot{\eta}^i \geqq 0 \tag{2.10}$$

for almost all x in $A(\beta)-B(\beta)$, and the differential equations

$$\psi^\rho_{y^i}\eta^i + \psi^\rho_{p^i}\dot{\eta}^i = 0 \tag{2.11}$$

for almost all x on x^1x^2, it will be called an *admissible variation*. For each admissible variation, the second variation

$$\begin{aligned} J_2(\eta) &= \int_{x^1}^{x^2} 2\omega(x, \eta, \dot{\eta})dx \\ &= \int_{x^1}^{x^2} (F_{y^iy^k}\eta^i\eta^k + 2F_{y^ip^k}\eta^i\dot{\eta}^k + F_{p^ip^k}\dot{\eta}^i\dot{\eta}^k)dx \end{aligned} \tag{2.12}$$

is well defined.

THEOREM 2.1. *Let C_0 be an arc of class C' which lies in $\mathfrak{D}$, satisfies the end conditions* (2.1), *and along which the matrix* (2.2) *has rank $m+t$. Suppose an admissible set of multipliers can be found for which $J_2(\eta)>0$ for every nonnull admissible variation. Then there is a neighborhood $\mathfrak{F}$ of C_0 in (x, y)-space such that the inequality $I(C)>I(C_0)$ holds for every admissible arc C in $\mathfrak{F}$ which lies in $\mathfrak{D}$, satisfies the end conditions* (2.1), *and is different from C_0.*

3. **The problem with a finite number of variables.** In order to be able to draw conclusions from the inequality (2.6) and also to furnish a model for the calculus of variations problem, we find it convenient to discuss first the problem of minimizing a function $f(x)$ of n variables x^i in the class of points x satisfying m inequalities $\phi^\beta(x) \geqq 0$ and t equations $\psi^\rho(x)=0$, when $m+t<n$. For the normal case, which is the only one we consider, our results are more general than those of Karush [7].

We suppose that the functions $f(x)$, $\phi^\beta(x)$, and $\psi^\rho(x)$ are of class C'' in a neighborhood $\mathfrak{R}$ of a point x_0, that $\phi^\beta(x_0) \geqq 0$, $\psi^\rho(x_0) = 0$ and that the matrix

$$(3.1) \qquad \left\| \begin{matrix} \phi^\beta_{x^i}(x_0) \\ \psi^\rho_{x^i}(x_0) \end{matrix} \right\|$$

has rank $m+t$.

THEOREM 3.1. *If x_0 minimizes $f(x)$ in the class of points x near x_0 such that $\phi^\beta(x) \geqq 0$, $\psi^\rho(x) = 0$, then there exist unique multipliers μ^β, μ^ρ such that if $F(x, \mu) = f(x) + \mu^\beta \phi^\beta(x) + \mu^\rho \psi^\rho(x)$, then $F_{x^i}(x_0, \mu) = 0$. Moreover, $\mu^\beta \leqq 0$ and $\mu^\beta \phi^\beta(x_0) = 0$ for each β.*

The first sentence of the theorem follows from well known results [1, p. 210] since the point x_0 is a normal point which minimizes $f(x)$ in the class of points x near x_0 for which $\phi^\beta(x) = \phi^\beta(x_0)$, $\psi^\rho(x) = 0$. To prove that $\mu^\beta \leqq 0$ we pick functions $\psi^j(x)$ $(j = m+t+1, \cdots, n)$ of class C'' such that the equations

$$(3.2) \qquad \phi^\beta(x) = v^\beta + \phi^\beta(x_0), \qquad \psi^\rho(x) = 0, \qquad \psi^j(x) = \psi^j(x_0) + v^j$$

have a nonvanishing functional determinant when $x = x_0$, $v^\beta = 0$, $v^j = 0$. Solutions $x^i(v)$ of class C'' near $v = 0$ therefore exist such that $x^i(0) = x_0^i$ and so the function

$$(3.3) \qquad f[x(v)] = F[x(v), \mu] - \mu^\beta v^\beta - \mu^\beta \phi^\beta(x_0)$$

is minimized by $v = 0$ in the class of points v near 0 for which each $v^\beta \geqq 0$. Its derivative with respect to v^β, namely $-\mu^\beta$, must therefore be non-negative. If $\phi^\beta(x_0) > 0$ for some β, then the function (3.3) has a two-sided minimum when regarded as a function of v^β only and so $\mu^\beta = 0$. Hence $\mu^\beta \phi^\beta(x_0) = 0$ for each β.

Let us define Γ as the set of all indices β such that $\phi^\beta(x_0) = 0$, and Δ as the set of all indices β such that $\mu^\beta < 0$. It is clear then that Δ is a subset of Γ.

THEOREM 3.2. *If x_0 minimizes $f(x)$ in the class of points x near x_0 such that $\phi^\beta(x) \geqq 0$, $\psi^\rho(x) = 0$, then the function $F(x, \mu)$ defined by Theorem 3.1 is such that*

$$F_{x^i x^k}(x_0, \mu) \pi^i \pi^k \geqq 0$$

for every set π^i such that

$$\psi^\rho_{x^i}(x_0)\pi^i = 0, \qquad \phi^\gamma_{x^i}(x_0)\pi^i \geqq 0, \qquad \phi^\delta_{x^i}(x_0)\pi^i = 0$$

for all ρ, all γ in $\Gamma - \Delta$, and all δ in Δ.

If π^i is a set as described in the theorem, we set $v^\beta(t) = \phi^\beta_{x^i}(x_0)\pi^i t$, $v^j(t) = \psi^j_{x^i}(x_0)\pi^i t$ and observe that the function $f[x\{v(t)\}]$, in which $x(v)$ is the function constructed in the proof of Theorem 3.1, is minimized by $t = 0$ in the class of non-negative t near $t=0$. The first derivative of this function obviously vanishes when $t=0$ and so its second derivative must be non-negative when $t=0$, or $F_{x^ix^k}(x_0, \mu)\dot{x}_0^i\dot{x}_0^k \geqq 0$, in which $\dot{x}_0^i = dx^i[v(t)]/dt$ when $t=0$. If we differentiate equations (3.3) after replacing v by $v(t)$, we discover that

$$\phi^\beta_{x^i}(x_0)\dot{x}_0^i = \phi^\beta_{x^i}(x_0)\pi^i, \qquad \psi^\rho_{x^i}(x_0)\dot{x}_0^i = 0 = \psi^\rho_{x^i}(x_0)\pi^i,$$
$$\psi^j_{x^i}(x_0)\dot{x}_0^i = \psi^j_{x^i}(x_0)\pi^i.$$

It follows that $\dot{x}_0^i = \pi^i$ and hence that $F_{x^ix^k}(x_0, \mu)\pi^i\pi^k \geqq 0$, as desired.

COROLLARY. *If x_0 minimizes $f(x)$ in the class of points x near x_0 such that $\phi^\beta(x) \geqq 0$, $\psi^\rho(x)=0$, and if the set $\Gamma-\Delta$ contains at most one index, γ, then the function $F(x, \mu)$ defined by Theorem 3.1 is such that $F_{x^ix^k}(x_0, \mu)\pi^i\pi^k \geqq 0$ for every set π^i such that $\psi^\rho_{x^i}(x_0)\pi^i=0$, $\phi^\delta_{x^i}(x_0)\pi^i=0$ for all ρ and all δ in Δ.*

There is nothing new to prove unless π^i is a set satisfying the conditions of the corollary for which $\phi^\gamma_{x^i}(x_0)\pi^i<0$. In this case we define $\bar{\pi}^i = -\pi^i$ and have a set $\bar{\pi}^i$ satisfying the conditions of Theorem 3.2. Since the quadratic form is unaltered when π^i is replaced by $\bar{\pi}^i$, we see that the corollary is true.

THEOREM 3.3. *Suppose there exist multipliers $\mu^0 \geqq 0$, $\mu^\beta \leqq 0$, μ^ρ such that if $F(x, \mu) = \mu^0 f(x) + \mu^\beta\phi^\beta(x) + \mu^\rho\psi^\rho(x)$, then $F_{x^i}(x_0, \mu)=0$, $\mu^\beta\phi^\beta(x_0)=0$ for each β. If the quadratic form $F_{x^ix^k}(x_0, \mu)\pi^i\pi^k>0$ for all nonnull π^i such that $\psi^\rho_{x^i}(x_0)\pi^i=0$, $\phi^\gamma_{x^i}(x_0)\pi^i \geqq 0$, $\phi^\delta_{x^i}(x_0)\pi^i=0$ for all ρ, all γ such that $\mu^\gamma=\phi^\gamma(x_0)=0$, and all δ such that $\mu^\delta<0$, then there is a neighborhood $\mathcal{J}$ of x_0 such that $f(x)>f(x_0)$ if x is any point in $\mathcal{J}$ different from x_0 for which $\phi^\beta(x) \geqq 0$, $\psi^\rho(x)=0$.*

If this theorem were false, there would be a sequence of points $x_r \neq x_0$ converging to x_0 such that $\phi^\beta(x_r) \geqq 0$, $\psi^\rho(x_r)=0$, $f(x_r) \leqq f(x_0)$. Define k_r as the positive square root of $|x_r-x_0|^2 - \mu^\beta\phi^\beta(x_r)$. Then k_r converges to zero. If we define $\pi_r^i = (x_r^i - x_0^i)/k_r$, then $|\pi_r| \leqq 1$ since $-\mu^\beta\phi^\beta(x_r) \geqq 0$. By passing to a subsequence we may therefore suppose that π_r^i converges to a limit π_0^i. Since $\mu^0 \geqq 0$, we have that

$$0 \geqq k_r^{-2}[F(x_r, \mu) - F(x_0, \mu) - \mu^\beta\phi^\beta(x_r)].$$

Expanding the right-hand side of this inequality by Taylor's theorem, making use of the fact that $F_{x^i}(x_0, \mu)=0$, and letting r approach ∞, we find that

$$(3.4) \qquad 0 \geqq \frac{1}{2}F_{x^ix^j}(x_0, \mu)\pi_0^i\pi_0^j + \limsup\,[-k_r^{-2}\mu^\beta\phi^\beta(x_r)].$$

Moreover, we see from Taylor's theorem and the relations

$$-\frac{k_r}{u^\delta} > \frac{\phi^\delta(x_r)}{k_r} \geqq 0, \qquad \frac{\phi^\gamma(x_r)}{k_r} \geqq 0, \qquad \frac{\psi^\rho(x_r)}{k_r} = 0,$$

that $\phi^\delta_{x^i}(x_0)\pi^i_0=0$ if $u^\delta<0$, $\phi^\gamma_{x^i}(x_0)\pi^i_0\geqq 0$ if $\phi^\gamma(x_0)=0$, and $\psi^\rho_{x^i}(x_0)\pi^i_0=0$. If the set π^i_0 were nonnull, it would then follow from the hypothesis of Theorem 3.3 that $F_{x^ix^j}(x_0)\pi^i_0\pi^j_0>0$ and so (3.4) could not hold since lim sup $[-k_r^{-2}\mu^\beta\phi^\beta(x_r)]\geqq 0$. Hence $\pi^i_0=0$. It then follows from (3.4) that

$$\lim k_r^{-2}\mu^\beta\phi^\beta(x_r) = 0. \tag{3.5}$$

Since we have from the definition of k_r that $1=|\pi_r|^2-k_r^{-2}\mu^\beta\phi^\beta(x_r)$, and since $\pi^i_0=0$, we see that (3.5) cannot hold. We infer the truth of the theorem from this contradiction.

4. **Some consequences of the hypotheses.** For a fixed x on x^1x^2 it follows from our hypotheses (2.4) and (2.6) that the function of q,

$$E_F(x, y_0, \dot y_0, q, \lambda) - bE_L(\dot y_0, q) - \lambda^\beta(x)\phi^\beta(x, y_0, q) + b\lambda^\beta(x)\bar\phi^\beta(x, y_0, q),$$

is minimized by $q^i=\dot y^i_0$ in the class of q^i near $\dot y^i_0$ for which $\phi^\beta(x, y_0, q)\geqq 0$, $\psi^\rho(x, y_0, q)=0$. From Theorem 3.1 we conclude that there exist multipliers μ^β, μ^ρ such that

$$[\mu^\beta - (1-b)\lambda^\beta]\phi^\beta_{p^i} + \mu^\rho\psi^\rho_{p^i} = 0$$

when $y^i=y^i_0(x)$, $p^i=\dot y^i_0(x)$. Since the matrix (2.2) has rank $m+t$, it follows that $\mu^\beta=(1-b)\lambda^\beta$, $\mu^\rho=0$ and since $0<b<1$, we also conclude from Theorem 3.1 that $\lambda^\beta\leqq 0$. From Theorem 3.3 we have that

$$\frac{\partial^2}{\partial q^i\partial q^k}[E_F(x, y_0, \dot y_0, q, \lambda) - bE_L(\dot y_0, q)]\pi^i\pi^k \geqq 0$$

when $q^i=\dot y^i_0(x)$ for all π^i such that $\psi^\rho_{p^i}\pi^i=0$, $\phi^\gamma_{p^i}\pi^i\geqq 0$, $\phi^\delta_{p^i}\pi^i=0$ for all ρ, all γ in $\Gamma(x)-\Delta(x)$ and all δ in $\Delta(x)$. Since (2.5) holds we have the following theorem.

THEOREM 4.1. *If the multipliers satisfy* (2.4) *and* (2.6), *then*

$$\lambda^\beta(x) \leqq 0. \tag{4.1}$$

If they also satisfy (2.5), *then* $F_{p^ip^k}\pi^i\pi^k\geqq bL_{p^ip^k}\pi^i\pi^k$ *holds for every set* π^i *for which* $\psi^\rho_{p^i}\pi^i=\phi^\delta_{p^i}\pi^i=0$ *for all* ρ *and all* δ *in the set* $\Delta(x)$ *for which* $\lambda^\delta(x)<0$.

COROLLARY 1. *If the multipliers satisfy* (2.4), (2.5), *and* (2.6), *then the matrix*

$$\left\|\begin{matrix} F_{p^ip^k} & \phi^\gamma_{p^i} & \psi^\rho_{p^i} \\ \phi^\gamma_{p^k} & 0 & 0 \\ \psi^\rho_{p^k} & 0 & 0 \end{matrix}\right\|, \tag{4.2}$$

in which γ *ranges over the set* Γ_r, *is nonsingular when* x *is in* $\bar A_r$, $y^i=y^i_0(x)$, $p^i=\dot y^i_0(x)$.

This result is an obvious corollary of Theorem **4.1** when it is remarked that $\Delta(x)$ is a subset of Γ_τ when x is in $\bar{A}_\tau$.

The following differentiability theorem is an immediate consequence of the preceding corollary.

COROLLARY 2. *When the multipliers are admissible, they and the functions* $\dot{y}_0^i(x)$ *are of class* C' *on each interval* $\bar{A}_\tau$.

COROLLARY 3. *There exists a constant* θ *such that if we define*

$$\overset{\beta}{\phi}_0(x, y, p) = \overset{\beta}{\lambda}(x)\overset{\beta}{\phi}(x, y, p),$$

for each β, *and*

$$H(x, y, p) = \theta[\overset{\rho}{\psi}(x, y, p)\overset{\rho}{\psi}(x, y, p) + \overset{\beta}{\phi}_0(x, y, p)\overset{\beta}{\phi}_0(x, y, p)],$$

then the quadratic form $(F_{p^ip^k}+H_{p^ip^k})\pi^i\pi^k>0$ *for all nonnull vectors* π^i *and each* x *on* x^1x^2.

This is a consequence of a known result [10, p. 679] since C_0 satisfies the differential equations $\psi^\rho(x, y, \dot{y})=0, \phi_0^\beta(x, y, \dot{y})=0$ and is such that $F_{p^ip^k}\pi^i\pi^k \geqq 0$ for all nonnull vectors π^i such that $\psi^\rho_{p^i}\pi^i=0$, $\phi^\beta_{0p^i}\pi^i=0$.

LEMMA 4.1. *There exist functions* $\psi^j(x, y, p)$ $(j=m+t+1, \cdots, n)$ *of class* C'' *near* C_0 *such that the determinant*

$$(4.3)\qquad \begin{vmatrix} \overset{\beta}{\phi}_{p^i} \\ \overset{\rho}{\psi}_{p^i} \\ \overset{j}{\psi}_{p^i} \end{vmatrix}$$

does not vanish on C_0.

This lemma is a well known consequence of the fact that the matrix (2.2) has rank $m+t$ along C_0 [1, pp. 224–226].

5. **The equivalence of** (2.6) **to** II_N **and condition** III′. We define the class $\mathcal{N}$ as the collection of all sets N of points (x, y, p, ν) such that there exists a neighborhood $\mathcal{R}_1$ of the points $(x, y_0, \dot{y}_0)$ on C_0 and a positive constant a for which (x, y, p, ν) is in N if and only if (x, y, p) is in $\mathcal{R}_1$, $\nu^0=\lambda^0$, $|\nu^\rho-\lambda^\rho(x)| \leqq a$, $|\nu^\beta-\lambda^\beta(x)| \leqq -a\lambda^\beta(x)$ if β is not in $\Gamma(x)-\Delta(x)$, $-a\leqq\nu^\gamma\leqq 0$ if γ is in $\Gamma(x)-\Delta(x)$. We shall say that the arc C_0 with the multipliers λ^0, $\lambda^\beta(x)$, $\lambda^\rho(x)$ satisfies the condition II_N if there is a set N in $\mathcal{N}$ such that

$$(5.1)\qquad E_F(x, y, p, q, \nu) - \nu^\beta\phi^\beta(x, y, q) \geqq 0$$

for all sets (x, y, p, q, ν) such that (x, y, p, ν) is in N, (x, y, p) and (x, y, q) are in $\mathcal{D}$ and $\phi^\beta(x, y, p)=0$ if β is in $\Delta(x)$.

We shall say that the arc C_0 with the multipliers λ^0, $\lambda^\beta(x)$, $\lambda^\rho(x)$ satisfies the condition III′ in case $F_{p^ip^k}[x, y_0(x), \dot{y}_0(x), \lambda(x)]\pi^i\pi^k>0$ for every nonnull vector π^i such that $\psi^\rho_{p^i}[x, y_0(x), \dot{y}_0(x)]\pi^i = \phi^\delta_{p^i}[x, y_0(x), \dot{y}_0(x)]\pi^i = 0$ for all ρ and all δ in $\Delta(x)$.

THEOREM 5.1. *If the arc C_0 with the multipliers λ^0, $\lambda^\beta(x)$, $\lambda^\rho(x)$ satisfies equations* (2.4), *then it satisfies the condition* II_N *and satisfies the condition* III′ *if and only if there is a constant b such that $0<b<1$ and a neighborhood $\mathfrak{D}_1$ of C_0 relative to the set $\mathfrak{D}$ such that the inequality* (2.6) *holds whenever (x, y, p) is in $\mathfrak{D}_1$, (x, y, q) is in $\mathfrak{D}$, and $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$.*

The proof of this theorem will follow from a series of lemmas which we now proceed to prove.

LEMMA 5.1. *If the arc C_0 with the multipliers λ^0, $\lambda^\beta(x)$, $\lambda^\rho(x)$ satisfies the condition* II_N *and equations* (2.4), *then $\lambda^\beta(x) \leqq 0$.*

This follows directly from Theorem 3.1 when we observe that

$$E_F(x, y_0, \dot{y}_0, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y_0, q) \geqq 0$$

for any q^i such that $\phi^\beta(x, y_0, q) \geqq 0$, $\psi^\rho(x, y_0, q) = 0$.

LEMMA 5.2. *If there is a positive constant b such that the inequality* (2.6) *holds for all (x, y, p) in a neighborhood $\mathfrak{D}_1$ of C_0 relative to $\mathfrak{D}$ for which $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$ and all (x, y, q) in $\mathfrak{D}$, and if C_0 with the multipliers λ^0, $\lambda^\beta(x)$, $\lambda^\rho(x)$ satisfies the condition* II_N, *then the constant b may be required to be less than one.*

If b is not less than one, it follows from Lemma 5.1 that if $b'<1$,

$$b[E_L(p, q) - \lambda^\beta(x)\bar{\phi}^\beta(x, y, q)] \geqq b'[E_L(p, q) - \lambda^\beta(x)\bar{\phi}^\beta(x, y, q)]$$

for all (x, y, q) in $\mathfrak{D}$. Hence the lemma is true.

LEMMA 5.3. *If the arc C_0 with the multipliers $\lambda^0 \geqq 0$, $\lambda^\beta(x) \leqq 0$, $\lambda^\rho(x)$ satisfies the condition* III′ *and satisfies* (2.4), *there exists a positive number b and a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ such that the inequality* (2.6) *holds for all (x, y, p) in $\mathfrak{D}_0$ such that $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$ and all (x, y, q) in $\mathfrak{D}_0$.*

The lemma may be proved by minor adaptations of the proof of Theorem 3.3.

We can easily infer from the identity

$$\begin{aligned} E_F(x, y, p, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, q) &= E_F(x, y, p, q, \nu) \\ &\quad - \nu^\beta\phi^\beta(x, y, q) + (\lambda^\rho - \nu^\rho)E_{\psi^\rho} + (\lambda^\beta - \nu^\beta)(E_{\phi^\beta} - \phi^\beta) \end{aligned} \tag{5.2}$$

and the definition of the class $\mathfrak{N}$ that the following lemma is true.

LEMMA 5.4. *If the arc C_0 with the multipliers λ^0, $\lambda^\beta(x)$, $\lambda^\rho(x)$ satisfies the condition* II_N, *there is a neighborhood $\mathfrak{D}_1$ of C_0 relative to the set $\mathfrak{D}$ and a positive constant a such that*

$$E_F(x, y, p, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, q) \geqq a \left| E_{\psi^\rho}(x, y, p, q) \right| \tag{5.3}$$

whenever (x, y, p) *is in* $\mathcal{D}_1$, $\phi^\beta(x, y, p) = 0$ *if* β *is in* $\Delta(x)$ *and* (x, y, q) *is in* $\mathcal{D}$. *Moreover, the set* $\mathcal{D}_1$ *can be chosen so that for each closed subset* M_γ *of* $B(\gamma)$ *there is a positive constant* a_γ *such that*

$$\begin{aligned} E_F(x, y, p, q, \lambda) &- \lambda^\beta(x)\phi^\beta(x, y, q) \\ &\geqq a_\gamma \left| E_{\phi^\gamma}(x, y, p, q) - \phi^\gamma(x, y, q) \right| \quad (\gamma \text{ not summed}) \end{aligned} \tag{5.4}$$

whenever (x, y, p) *is in* $\mathcal{D}_1$, $\phi^\beta(x, y, p) = 0$ *if* β *is in* $\Delta(x)$, *and* (x, y, q) *is in* $\mathcal{D}$, *and* x *is in* M_γ.

The proof of Theorem 5.1 may now be constructed by modifying the method used by Hestenes to prove a similar theorem [3, Theorem 4.3]. If a constant b and a neighborhood $\mathcal{D}_1$ having the properties described in Theorem 5.1 cannot be found, it follows from Lemmas 5.2 and 5.3 that there exists a sequence (x_k, y_k, p_k, q_k) such that (x_k, y_k, p_k) is in $\mathcal{D}$, (x_k, y_k, q_k) is in $\mathcal{D} - \mathcal{D}_0$, $\phi^\beta(x_k, y_k, p_k) = 0$ if β is in $\Delta(x_k)$,

$$\begin{aligned} E_F[x_k, y_k, p_k, q_k, \lambda(x_k)] &- \lambda^\beta(x_k)\phi^\beta(x_k, y_k, q_k) \\ &\leqq k^{-1}[E_L(p_k, q_k) - \lambda^\beta(x_k)\bar{\phi}^\beta(x_k, y_k, q_k)], \end{aligned} \tag{5.5}$$

and for which (x_k, y_k, p_k) converges to a point $(x_0, y_0, \dot{y}_0)$ on C_0. Since $\lambda^\beta(x)$ is continuous, we may suppose that

$$\phi^\beta(x_k, y_k, p_k) = 0$$

if β is in $\Delta(x_0)$.

CASE I. q_k has a finite accumulation point q_0. Then $q_0^i \neq \dot{y}_0^i$ for all i since $(x_0, y_0, \dot{y}_0)$ is on C_0 and (x_0, y_0, q_0) is not in $\mathcal{D}_0$. It follows from (2.7) that

$$E_L(p, q) \leqq 2L(q), \tag{5.6}$$

and since $\bar{\phi}^\beta(x, y, q) \leqq 1$, it follows from (5.5) and the condition II_N that

$$\lim \{E_F[(x_k, y_k, p_k, q_k, \lambda(x_k)] - \lambda^\beta(x_k)\phi^\beta(x_k, y_k, q_k)\} = 0. \tag{5.7}$$

Suppose for the moment that the set $\Gamma(x_0) - \Delta(x_0)$ is void. By Lemma 4.1 the equations

$$\begin{aligned} &\phi^\delta(x, y, r) = 0 \text{ if } \delta \text{ is in } \Delta(x_0), \quad \phi^\alpha(x, y, r) = \phi^\alpha(x, y, p) + v^\alpha \\ &\text{if } \alpha \text{ is not in } \Gamma(x_0), \ \psi^\rho(x, y, r) = 0, \quad \psi^j(x, y, r) = \psi^j(x, y, p) + v^j \end{aligned} \tag{5.8}$$

have solutions $r^i(x, y, p, v)$ of class C' near $(x_0, y_0, \dot{y}_0, 0)$ such that $r^i(x_0, y_0, \dot{y}_0, 0) = \dot{y}_0^i$, (x, y, r) is in $\mathcal{D}$, and $\phi^\delta(x, y, r) = 0$ if δ is in $\Delta(x)$. If we set $r_k^i(v) = r^i(x_k, y_k, p_k, v)$, it follows from the condition II_N that for $|v|$ sufficiently small,

$$\liminf \{E_F[x_k, y_k, r_k(v), q_k, \lambda(x_k)] - \lambda^\beta(x_k)\phi^\beta(x_k, y_k, q_k)\} \geqq 0,$$

and hence from (5.7)

$$\liminf \{E_F[x_k, y_k, r_k(v), q_k, \lambda(x_k)] - E_F[x_k, y_k, p_k, q_k, \lambda(x_k)]\} \geqq 0,$$

$$\begin{aligned} F[x_0, y_0, \dot{y}_0, \lambda(x_0)] - \dot{y}_0^i F_{p^i}[x_0, y_0, \dot{y}_0, \lambda)(x_0)] \\ - F[x_0, y_0, r_0(v), \lambda(x_0)] + r_0^i(v) F_{p^i}[x_0, y_0, r_0(v), \lambda(x_0)] \\ - q_0^i\{F_{p^i}[x_0, y_0, r_0(v), \lambda(x_0)] - F_{p^i}[x_0, y_0, \dot{y}_0, \lambda(x_0)]\} = 0. \end{aligned} \tag{5.9}$$

Define $v^\alpha(e) = \phi^\alpha[x_0, y_0, \dot{y}_0 + e(q_0 - \dot{y}_0)] - \phi^\alpha(x_0, y_0, \dot{y}_0)$ if α is not in $\Gamma(x_0)$, $v^j(e) = \psi^j[x_0, y_0, \dot{y}_0 + e(q_0 - \dot{y}_0)] - \psi^j(x_0, y_0, \dot{y}_0)$, $r^i(e) = r_0^i v(e)$,

$$\begin{aligned} Q(e) = F[x_0, y_0, \dot{y}_0, \lambda(x_0)] - \dot{y}_0^i F_{p^i}[x_0, y_0, \dot{y}_0, \lambda(x_0)] \\ - F[x_0, y_0, r(e), \lambda(x_0)] + r^i(e) F_{p^i}[x_0, y_0, r(e), \lambda(x_0)] \\ - q_0^i\{F_{p^i}[x_0, y_0, r(e), \lambda(x_0)] - F_{p^i}[x_0, y_0, \dot{y}_0, \lambda(x_0)]\}. \end{aligned}$$

By (5.9) and the fact that $r^i(0) = \dot{y}_0^i$ we see that $Q(e) \geqq Q(0) = 0$ for all sufficiently small e. Hence $Q'(0) = 0$, or

$$(\dot{y}_0^i - q_0^i) F_{p^i p^k}[x_0, y_0, \dot{y}_0, \lambda(x_0)] \dot{r}^i(0) = 0. \tag{5.10}$$

Now it follows from (5.3) that $\lim |E_{\psi^\rho}(x_k, y_k, p_k, . q_k)| = 0$. Since $\psi^\rho(x_k, y_k, p_k) = \psi^\rho(x_k, y_k, q_k) = 0$, it follows that

$$(q_0^i - \dot{y}_0^i) \psi_{p^i}^\rho(x_0, y_0, \dot{y}_0) = 0. \tag{5.11}$$

Similarly, it follows from (5.4) and the fact that $\phi^\delta(x_k, y_k, p_k) = 0$ if δ is in $\Delta(x_0)$ that

$$(q_0^i - \dot{y}_0^i) \phi_{p^i}^\delta(x_0, y_0, \dot{y}_0) = 0 \tag{5.12}$$

if δ is in $\Delta(x_0)$. If we differentiate the equations satisfied by $r(e)$ and set $e = 0$, we find that $\dot{r}^i(0) = q_0^i - \dot{y}_0^i$ and hence

$$F_{p^i p^k}[x_0, y_0, \dot{y}_0, \lambda(x_0)] \dot{r}^i(0) \dot{r}^k(0) = 0$$

although the numbers $\dot{r}^i(0)$ do not all vanish and satisfy

$$\psi_{p^i}^\rho(x_0, y_0, \dot{y}_0) \dot{r}^i(0) = \phi_{p^i}^\delta(x_0, y_0, \dot{y}_0) \dot{r}^i(0) = 0$$

for all ρ and all δ in $\Delta(x_0)$, and this is a contradiction of condition III'.

If the set $\Gamma(x_0) - \Delta(x_0)$ is not void, it contains a unique index γ. We show first that $\phi_{p^i}^\gamma(x_0, y_0, \dot{y}_0)(q_0^i - \dot{y}_0^i) \geqq 0$. We observe that if we set $\nu^\rho = \lambda^\rho$, $\nu^\beta = \lambda^\beta$ when $\beta \neq \gamma$, it follows from the identity (5.2) and the condition II_N that if $\nu^\gamma = -a$, then the point (x_k, y_k, p_k, ν) is in N for k sufficiently large, and

$$\begin{aligned} E_F[x_k, y_k, p_k, q_k, \lambda(x_k)] - \lambda^\beta(x_k) \phi^\beta(x_k, y_k, q_k) \\ \geqq - [a + \lambda^\gamma(x_k)][\phi^\gamma(x_k, y_k, p_k) + (q_k^i - p_k^i) \phi_{p^i}^\gamma(x_k, y_k, p_k)]. \end{aligned} \tag{5.13}$$

Since (5.7) holds and $\lambda^\gamma(x_0)=\phi^\gamma(x_0, y_0, \dot{y}_0)=0$, we find that

$$0 \geqq - a(q_0^i - \dot{y}_0^i)\phi^\gamma_{p^i}(x_0, y_0, \dot{y}_0).$$

Since $a>0$, we see that $\phi^\gamma_{p^i}(x_0, y_0, \dot{y}_0)(q_0^i-\dot{y}_0^i)\geqq 0$.

Suppose $\phi^\gamma_{p^i}(x_0, y_0, \dot{y}_0)(q_0^i-\dot{y}_0^i)=0$. Then we include the equation $\phi^\gamma(x, y, r)=0$ with the equations (5.8) and the analysis in the preceding paragraphs is unaltered since $\phi^\gamma_{p^i}(x_0, y_0, \dot{y}_0)\dot{r}^i(0)=0$. If $\phi^\gamma_{p^i}(x_0, y_0, \dot{y}_0)(q_0^i-\dot{y}_0^i)>0$, we include with the equations (5.8) the equation $\phi^\gamma(x, y, r)=v^\gamma$. Then the solutions $r^i(x, y, p, v)$ will be such that (x, y, r) will be in $\mathcal{D}$ if $v^\gamma\geqq 0$. We set $v^\gamma=\phi^\gamma[x_0, y_0, \dot{y}_0+e(q_0-\dot{y}_0)]$, the v^α and v^i being defined as before. Then $v^\gamma(e)\geqq 0$ if e is positive and sufficiently small and so $Q(e)\geqq Q(0)$ for all nonnegative small e. Hence $Q'(0)\geqq 0$, or $(\dot{y}_0^i-q_0^i)F_{p^ip^k}[x_0, y_0, \dot{y}_0, \lambda(x_0)]\dot{r}^i(0)\geqq 0$. It follows just as above from (5.11) and (5.12) that $\dot{r}^i(0)=q_0^i-\dot{y}_0^i$ and that the last inequality cannot hold when C_0 satisfies condition III′.

This disposes of the case in which q_k has a finite accumulation point. Next consider:

CASE II. $L(q_k)\to+\infty$. Then the sequences $q_k^i/L(q_k)$ converge to limits π^i and $\pi^i\pi^i=1$. It follows from (5.5) and (5.6) that

$$(5.14)\quad \lim \{E_F[x_k, y_k, p_k, q_k, \lambda(x_k)] - \lambda^\beta(x_k)\phi^\beta(x_k, y_k, q_k)\}/L(q_k) = 0.$$

If the set $\Gamma(x_0)-\Delta(x_0)$ is void, we define the functions $r_k^i(v)$ as in the bounded case. Instead of (5.9) we now get

$$- \pi^i\{F_{p^i}[x_0, y_0, r_0(v), \lambda(x_0) - F_{p^i}[x_0, y_0, \dot{y}_0, \lambda(x_0)]\} \geqq 0.$$

We define $v^\alpha(e)=\phi^\alpha(x_0, y_0, \dot{y}_0+e\pi)-\phi^\alpha(x_0, y_0, \dot{y}_0)$, $v^j(e)=\psi^j(x_0, y_0, \dot{y}+e\pi)-\psi^j(x_0, y_0, \dot{y}_0)$, $r^i(e)=r_0^i[v(e)]$, $Q(e)=-\pi^i\{F_{p^i}[x_0, y_0, r(e), \lambda(x_0)]-F_{p^i}[x_0, y_0, \dot{y}_0, \lambda(x_0)]\}$. Then $Q(e)\geqq Q(0)=0$ for all sufficiently small e, $Q'(0)=0$, and

$$(5.15)\quad - \pi^iF_{p^ip^k}[x_0, y_0, \dot{y}_0, \lambda(x_0)]\dot{r}^k(0) = 0.$$

Instead of (5.11) and (5.12) we infer from (5.14), (5.3), and (5.4) that $\psi^\rho_{p^i}(x_0, y_0, \dot{y}_0)\pi^i=\phi^\delta_{p^i}(x_0, y_0, \dot{y}_0)\pi^i=0$ for all ρ and all δ in $\Delta(x_0)$, and this with (5.15) contradicts the nonsingularity hypothesis, since it is easy to see by differentiating the equations satisfied by $r(e)$ that $\dot{r}^i(0)=\pi^i$. If there is an index γ in $\Gamma(x_0)-\Delta(x_0)$, we see from (5.13) and (5.14) that $\phi^\gamma_{p^i}(x_0, y_0, \dot{y}_0)\pi^i\geqq 0$. The modifications in the analysis for the bounded case can be easily carried out since this inequality holds.

To prove the converse of Theorem 5.1 we utilize the following lemma which is similar to the Corollary of Hestenes [3, Corollary 1, p. 59].

LEMMA 5.5. *There is a neighborhood $\mathcal{D}_0$ of C_0 relative to $\mathcal{D}$ and a positive constant b_1 such that*

$$(5.16)\quad E_{\phi^\beta}(x, y, p, q) - \phi^\beta(x, y, q) \leqq b_1E_L(p, q)$$

whenever (x, y, p) *is in* $\mathcal{D}_0$ *and* (x, y, q) *is in* $\mathcal{D}$. *Moreover, if* $H(x, y, p)$ *is of class* C'' *on* $\mathcal{R}$ *and is such that there exists a positive constant* W *and a neighborhood* $\mathcal{J}_1$ *of* C_0 *in* (x, y)*-space such that* $WL(p) \geqq |H(x, y, p)|$ *whenever* (x, y) *is in* $\mathcal{J}_1$ *and* (x, y, p) *is in* $\mathcal{D}$, *then* $\mathcal{D}_0$ *and* b_1 *may be chosen so that*

$$(5.17) \qquad |E_H(x, y, p, q)| \leqq b_1 E_L(p, q)$$

whenever (x, y, p) *is in* $\mathcal{D}_0$ *and* (x, y, q) *is in* $\mathcal{D}$.

Since $L_{p^ip^k}\pi^i\pi^k > 0$ for all nonnull sets π^i, it follows from reasoning like that used in Lemma 5.3 that there is a neighborhood $\mathcal{R}_1$ of C_0 in (x, y, p)-space and a positive constant b_2 such that (5.17) holds with b_1 replaced by b_2 whenever (x, y, p) and (x, y, q) are in $\mathcal{R}_1$ for both H and ϕ^β. Since $\phi^\beta \geqq 0$ on $\mathcal{D}$, (5.16) also holds with b_1 replaced by b_2 whenever (x, y, p) and (x, y, q) are in $\mathcal{R}_1\mathcal{D}$. Choose a neighborhood $\mathcal{D}_0$ of C_0 relative to $\mathcal{D}$ such that the closure of $\mathcal{D}_0$ is in $\mathcal{R}_1$. Then there exists a positive constant b_3 such that $L(q) \leqq b_3 E_L(p, q)$ whenever (x, y, p) is in $\mathcal{D}_0$ and (x, y, q) is in $\mathcal{D} - \mathcal{R}_1\mathcal{D}$. Let b_4 be an upper bound for $|H(x, y, p) - p^i H_{p^i}(x, y, p)|$, $|H_{p^i}(x, y, p)|$, $|\phi^\beta(x, y, p) - p^i\phi^\beta_{p^i}(x, y, p)|$, $|\phi^\beta_{p^i}(x, y, p)|$ whenever (x, y, p) is in $\mathcal{D}_0$. If (x, y, p) is in $\mathcal{D}_0$ and (x, y, q) is in $\mathcal{D} - \mathcal{R}_1\mathcal{D}$, then

$$|E_H(x, y, p, q) - H(x, y, q)| \leqq b_4(n + 1)L(q) \leqq b_3 b_4(n + 1)E_L(p, q),$$

and the same inequality holds if H is replaced by ϕ^β. It follows that (5.16) is true if $b_1 = b_2 + b_3 b_4(n+1)$ whenever (x, y, p) is in $\mathcal{D}_0$ and (x, y, q) is in $\mathcal{D}$. Reduce $\mathcal{D}_0$ if necessary so that (x, y) is in $\mathcal{J}_1$ if (x, y, p) is in $\mathcal{D}_0$. Then it is clear that (5.17) holds if $b_1 = b_2 + Wb_3 + b_3 b_4(n+1)$ whenever (x, y, p) is in $\mathcal{D}_0$ and (x, y, q) is in $\mathcal{D}$.

We may now complete the proof of Theorem 5.1. Suppose a positive constant b and a neighborhood $\mathcal{R}_1$ of C_0 in (x, y, p) space exist such that the inequality (2.6) holds whenever (x, y, p) is in $\mathcal{R}_1\mathcal{D}$, (x, y, q) is in $\mathcal{D}$, and $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$. By Lemma 5.5 we may find a positive constant b_1 and reduce $\mathcal{D}_1$ if necessary so that $|E_{\psi^\rho}| \leqq b_1 E_L$, $|E_{\bar\phi^\beta}| \leqq b_1 E_L$, $E_{\phi^\beta} - \phi^\beta \leqq b_1 E_L$ if (x, y, p) is in $\mathcal{R}_1\mathcal{D}$ and (x, y, q) is in $\mathcal{D}$. Define N as the member of the class $\mathcal{N}$ determined by the neighborhood $\mathcal{R}_1$ and a positive number a such that $a < b$, $ab_1[t+1+m \max |\lambda^\beta(x)|] < b$. If (x, y, p, q, ν) is such that (x, y, p, ν) is in N, (x, y, p) and (x, y, q) are in $\mathcal{D}$, and $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$, then $\nu^\gamma(E_{\phi^\gamma} - \phi^\gamma) \geqq -ab_1 E_L$ if γ is in $\Gamma(x) - \Delta(x)$,

$$(\nu^\alpha - \lambda^\alpha)E_{\bar\phi^\alpha} \geqq -ab_1 \max |\lambda^\alpha(x)| E_L, \qquad (\nu^\alpha - \lambda^\alpha)\bar\phi^\alpha \geqq a\lambda^\alpha(x)\bar\phi^\alpha,$$

$(\nu^\rho - \lambda^\rho)E_{\psi^\rho} \geqq -ab_1 E_L$. Now if α is in $\Delta(x)$, then $\phi^\alpha(x, y, p) = \bar\phi^\alpha(x, y, p) = 0$, $\phi^\alpha_{p^i}(x, y, p) = \bar\phi^\alpha_{p^i}(x, y, p)$, while if α is not in $\Gamma(x)$, then $\nu^\alpha = \lambda^\alpha = 0$. Hence $(\nu^\alpha - \lambda^\alpha)(E_{\phi^\alpha} - \phi^\alpha) = (\nu^\alpha - \lambda^\alpha)(E_{\bar\phi^\alpha} - \bar\phi^\alpha)$ if α is restricted to the set complementary to $\Gamma(x) - \Delta(x)$. It follows from these relations and the identity (5.2), written in the form

$$E_F(x, y, p, q, \nu) - \nu^\beta\phi^\beta(x, y, q) = E_F(x, y, p, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, q)$$
$$+ (\nu^\rho - \lambda^\rho)E_{\psi^\rho} + \nu^\gamma(E_{\phi^\gamma} - \phi^\gamma) + (\nu^\alpha - \lambda^\alpha)E_{\bar\phi^\alpha} - (\nu^\alpha - \lambda^\alpha)\bar\phi^\alpha,$$

that the left-hand side of (5.1) is not less than

$$b[E_L(p, q) - \lambda^\beta(x)\bar\phi^\beta(x, y, q)] - atb_1E_L(p, q) - ab_1E_L(p, q)$$
$$- a\max|\lambda^\alpha(x)|mb_1E_L(p, q) + a\lambda^\alpha(x)\bar\phi^\alpha(x, y, q) \geqq 0$$

by virtue of our choice of a. This completes the proof of Theorem 5.1 since III′ follows from Theorem 4.1.

6. **Convergent sequences of admissible arcs.** In the proof of Theorem 2.1 we shall need to be able to draw conclusions on the convergence of the derivatives $\dot y_r^i$ of a sequence of admissible arcs C_r which converge to C_0 uniformly in (x, y)-space. The particular results needed can be deduced from the following theorem.

THEOREM 6.1. *Let C_0 satisfy the hypotheses of Theorem 2.1. If C_r is a sequence of admissible arcs in $\mathfrak{D}$ which converge uniformly to C_0 in (x, y)-space such that* $\limsup J(C_r) \leqq J(C_0)$, *then it is true that* $\lim K(C_r, C_0) = 0$, *and that there is a subsequence C_{r_k} of the sequence C_r such that* $\lim \dot y_{r_k}^i = \dot y_0^i$ *almost uniformly on x^1x^2.*

Here $J(C)$ and $K(C, C_0)$ are defined as

$$J(C) = \int_{x^1}^{x^2} [F(x, y, \dot y, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, \dot y)]dx,$$

$$K(C, C_0) = \int_{x^1}^{x^2} [L(\dot y - \dot y_0) - 1]dx.$$

Consider the equations

$$\phi^\beta(x, y, P) = \phi^\beta(x, y_0, \dot y_0), \qquad \psi^\rho(x, y, P) = 0, \tag{6.1}$$
$$\psi^j(x, y, P) = \psi^j(x, y_0, \dot y_0)$$

in which the functions ψ^j are those of Lemma 4.1. By Corollary 2 to Theorem 4.1 the functions involved in equations (6.1) are of class C' for x on each $\bar A_r$ and for (x, y, P) near C_0. Hence there exist solutions $P^i(x, y)$ of class C' if x is in $\bar A_r$ and y is near $y_0(x)$ such that $P^i[x, y_0(x)] = \dot y_0^i(x)$. Since $\dot y_0^i(x)$ is continuous, it follows that equations (6.1) are satisfied when x is an end point x_r of an interval A_r by $P^i(x_r^-, y)$ as well as by $P^i(x_r^+, y)$. Since the equations (6.1) admit only one solution near $\dot y_0^i(x)$ when y^i is sufficiently near $y_0^i(x)$, it follows that $P^i(x_r^-, y) = P^i(x_r^+, y)$, and hence that $P^i(x, y)$ is continuous on the whole interval x^1x^2. Differentiating this last equation we see also that $P_{y^k}^i(x, y)$ is continuous on x^1x^2.

With the help of the functions $P^i(x, y)$, $J(C)$ can be written as the sum

(6.2) $$J(C) = J^*(C) + E^*(C),$$

in which

$$J^*(C) = \int_{x^1}^{x^2} [F(x, y, P, \lambda) + (\dot{y}^i - P^i)F_{p^i}(x, y, P, \lambda)]dx,$$

$$E^*(C) = \int_{x^1}^{x^2} [E_F(x, y, P, \dot{y}, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, \dot{y})]dx.$$

For any function $H(x, y, p)$ of class C'' on $\mathfrak{D}$ we define

(6.3) $$H(C) = \int_{x^1}^{x^2} H(x, y, \dot{y})dx.$$

We shall be interested only in the case in which there is a positive constant c and a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ such that

(6.4) $$E_F(x, y, p, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, q) \geqq c \left| E_H(x, y, p, q) \right|$$

whenever (x, y, p) is in $\mathfrak{D}_0$, (x, y, q) is in $\mathfrak{D}$, and $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$. We shall say that H is E^*-dominated by F near C_0 on $\mathfrak{D}$ when this is true.

We shall also restrict the constant c and the neighborhood $\mathfrak{D}_0$ so that

(6.5) $$E_F(x, y, p, q, \lambda) - \lambda^\beta(x)\phi^\beta(x, y, q) = cE_L(p - P, q - P)$$

whenever (x, y, p) is in $\mathfrak{D}_0$, (x, y, q) is in $\mathfrak{D}$, and $\phi^\beta(x, y, p) = 0$ if β is in $\Delta(x)$. The possibility of doing this follows from the E^*-dominance of L by F near C_0 on $\mathfrak{D}$ and from Lemma 5.5 when we observe that $L(p-P)$ satisfies the hypotheses imposed on $H(x, y, p)$ in that Lemma.

Theorem 6.1 will be a consequence of the following lemma, whose proof is identical with that of a similar lemma of Hestenes [4, Theorem 5.1].

LEMMA 6.1. *Let C_0 satisfy the hypotheses of Theorem* 2.1. *Given a constant* $\epsilon > 0$, *there exists a constant* $\eta > 0$ *and a neighborhood* $\mathfrak{J}$ *of* C_0 *in* (x, y)*-space such that the inequality*

$$H(C) - H(C_0) < \epsilon$$

holds for every admissible arc C in $\mathfrak{J}$ which lies in $\mathfrak{D}$, satisfies the end conditions (2.1), *and is such that*

$$J(C) \leqq J(C_0) + \eta.$$

The first part of Theorem 6.1 will follow from Lemma 6.1 provided that $H(C) = K(C, C_0)$ has an integrand $H(x, y, p) = L(p - \dot{y}_0) - 1$ which is E^*-dominated by F near C_0 on $\mathfrak{D}$. Since this can be shown just as (6.5) was shown, we conclude that

$$\lim K(C_r, C_0) = 0.$$

To prove the second part of the theorem, we observe that Schwarz' inequality implies that

$$\left\{\int_{x^1}^{x^2} |\dot{y}_r - \dot{y}_0|\,dx\right\}^2 = \left\{\int_{x^1}^{x^2} [L(\dot{y}_r - \dot{y}_0) - 1]^{1/2}[L(\dot{y}_r - \dot{y}_0) + 1]^{1/2}dx\right\}^2$$

$$\leqq K(C_r, C_0)\int_{x^1}^{x^2} [L(\dot{y}_r - \dot{y}_0) + 1]dx$$

$$= K(C_r, C_0)\left[2(x^2 - x^1) + K(C_r, C_0)\right].$$

It follows that $\dot{y}_r^i$ converges in mean of order one to $\dot{y}_0^i$. The existence of a subsequence which converges almost everywhere to $\dot{y}_0^i$ (and hence almost uniformly to $\dot{y}_0^i$) is a well known consequence of convergence in mean [2, Theorem 23, p. 242 and Theorem 19, p. 239].

7. **The variation η_0^i.** Our proof of Theorem 2.1 is indirect. Suppose the hypotheses of Theorem 2.1 are fulfilled but that the conclusion is not. Then there is a sequence C_r: $y^i = y_r^i(x)$ of curves in $\mathfrak{D}$ which satisfy the end conditions (2.1), are different from C_0, and such that

$$\lim y_r^i(x) = y_0^i(x)$$

uniformly on x^1x^2, and yet $I(C_r) \leqq I(C_0)$. Since C_r and C_0 are in $\mathfrak{D}$ and $\lambda^0 \geqq 0$, $J(C_r) = \lambda^0 I(C_r) \leqq \lambda^0 I(C_0) = J(C_0)$. By virtue of Theorem 6.1 we may replace C_r by a subsequence for which $\dot{y}_r^i$ converges almost uniformly to $\dot{y}_0^i$ on x^1x^2 and hence for which $\lim k_r = 0$, in which

$$k_r \geqq 0, \qquad k_r^2 = K(C_r, C_0) - \int_{x^1}^{x^2} \lambda^\beta(x)\Phi^\beta(x, y_r, \dot{y}_r)dx.$$

This follows since $\lambda^\beta(x)\Phi^\beta(x, y_r, \dot{y}_r)$ converges boundedly to $\lambda^\beta(x)\Phi^\beta(x, y_0, \dot{y}_0) = 0$ almost everywhere on x^1x^2. Let us define

$$\eta_r^i(x) = (y_r^i - y_0^i)/k_r.$$

Then it is clear that

$$\int_{x^1}^{x^2} \left\{ |\dot{\eta}_r|^2/h_r(x)\right\}dx \leqq 1, \tag{7.1}$$

in which

$$h_r(x) = 1 + L(\dot{y}_r - \dot{y}_0) = k_r^2|\dot{\eta}_r|^2/[L(\dot{y}_r - \dot{y}_0) - 1].$$

LEMMA 7.1. *The integrals of the functions $h_r(x)$ are absolutely continuous uniformly with respect to r.*

This follows since $h_r(x)$ differs by 2 from the integrand of $K(C_r, C_0)$ and $K(C_r, C_0)$ tends to zero.

LEMMA 7.2. *The functions* $\eta_r^i(x)$ *are absolutely continuous uniformly with respect to* r.

By Schwartz's inequality and (7.1),

$$\left|\int_M \dot{\eta}_r^i dx\right|^2 \leqq \int_M (|\dot{\eta}_r|^2/h_r)dx \int_M h_r dx \leqq \int_M h_r dx. \tag{7.2}$$

Thus the result follows from Lemma 7.1.

LEMMA 7.3. *The sequence of arcs* C_r *may be chosen so that there exists a function* $\eta_0^i(x)$ *satisfying the end conditions* (2.8) *and such that* $\lim \eta_r^i(x) = \eta_0^i(x)$ *uniformly on* x^1x^2. *Moreover,* $\eta_0^i(x)$ *is absolutely continuous and*

$$\int_{x^1}^{x^2} |\dot{\eta}_0|^2 dx \leqq 2.$$

By Lemma 7.2 the functions $\eta_r^i(x)$ are absolutely continuous uniformly in r. Since $\eta_r^i(x^1) = 0$, the functions $\eta_r^i(x)$ are also uniformly bounded. By Ascoli's theorem [2, p. 122] subsequences can be found which converge uniformly to limits $\eta_0^i(x)$ and these limits are obviously absolutely continuous and such that $\eta_0^i(x^s) = 0$.

The proof of the integrability of $|\dot{\eta}_0|^2$ to an integral bounded by two is identical with that of the corresponding assertion when there are no differential inequalities [6, pp. 528–529].

LEMMA 7.4. *If* $g(x)$ *is bounded and measurable, and if* $N_{ir}(x)$ *are continuous functions which converge uniformly to* $N_{i0}(x)$ *on* x^1x^2, *then*

$$\lim \int_M g(x)(\eta_r^i - \eta_0^i)dx = 0, \tag{7.3}$$

$$\lim \int_M N_{ir}(x)\dot{\eta}_r^i dx = \int_M N_{i0}(x)\dot{\eta}_0^i dx \tag{7.4}$$

for every measurable subset M *of* x^1x^2. *If* $|g(x)|^2$ *is integrable, then*

$$\lim \int_M g(x)(\dot{\eta}_r^i - \dot{\eta}_0^i)dx = 0 \tag{7.5}$$

for every measurable subset M *of* x^1x^2 *on which* $\dot{y}_r^i$ *converges uniformly.*

The relation (7.3) is obvious since $\eta_r^i(x)$ converges uniformly to $\eta_0^i(x)$. If $|g(x)|^2$ is integrable, there exists for each $\epsilon > 0$ a bounded function $G(x)$ such that [8, p. 229]

$$\int_{x^1}^{x^2} |g(x) - G(x)|^2 dx < \epsilon.$$

If M is a set on which $\dot{y}_r^i$ converges uniformly to $\dot{y}_0^i$ on M, then $h_r(x)$ converges uniformly to 2 on M and so it follows from (7.1) and Lemma 7.3 that

$$\int_M |\dot{\eta}_r|^2 dx \leqq 3, \qquad \int_M |\dot{\eta}_0|^2 dx \leqq 2$$

if r is sufficiently large. From Schwarz' inequality we then have

$$\left| \int_M [g(x) - G(x)](\dot{\eta}_r^i - \dot{\eta}_0^i) dx \right|^2 \leqq 2 \int_M [g(x) - G(x)]^2 dx \int_M (|\dot{\eta}_r|^2 + |\dot{\eta}_0|^2) dx \leqq 10\epsilon.$$

To prove (7.5) it is therefore sufficient to prove it when $g(x)$ is bounded. The relation (7.5) when $g(x)$ is bounded and M is any measurable subset of x^1x^2 and the relation (7.4) follow from known results on Lebesgue-Stieltjes integrals [see 2, Theorem 28, p. 285; Theorem 21, p. 280; Corollary to Theorem 20, p. 280].

8. **Some auxiliary functions.** Let us recall from §6 the definition of the functions $P^i(x, y)$ such that $[x, y, P(x, y)]$ lies in $\mathfrak{D}$ whenever (x, y) is in a neighborhood of C_0 and such that $P^i(x, y_0) = \dot{y}_0^i$. Define $p_r^i(x) = P^i[x, y_r(x)]$, $\pi_r^i(x) = [p_r^i(x) - p_0^i(x)]/k_r$, $\pi_0^i(x) = P_{y^v}^i[x, y_0(x)]\eta_0^v(x)$.

LEMMA 8.1. *The following relations hold:*

$$(8.1) \qquad \phi^\beta(x, y_r, p_r) = \phi^\beta(x, y_0, \dot{y}_0), \qquad \psi^\rho(x, y_r, p_r) = 0,$$

$$(8.2) \qquad \lim p_r^i(x) = p_0^i(x) = \dot{y}_0^i(x) \textit{ uniformly on } x^1x^2,$$

$$(8.3) \qquad \lim \pi_r^i(x) = \pi_0^i(x) \textit{ uniformly on } x^1x^2.$$

Equations (8.1) are immediate consequences of the definitions. Equations (8.2) follow from the uniform convergence of $y_r^i(x)$ to $y_0^i(x)$ and the continuity of $P^i(x, y)$ for x on x^1x^2 and y near $y_0(x)$. To prove equations (8.3) observe that Taylor's formula yields

$$\pi_r^i(x) = B_{vr}^i(x)\eta_r^v(x),$$

in which

$$B_{vr}^i(x) = \int_0^1 P_{y^v}^i[x, y_0 + \theta(y_r - y_0)]d\theta.$$

Since y_r^i converges uniformly to y_0^i and since $P_{y^v}^i(x, y)$ is continuous for x on x^1x^2 and y near $y_0(x)$, $B_{vr}^i(x)$ converges uniformly to $P_{y^v}^i[x, y_0(x)]$. By Lemma 7.3, $\eta_r^v(x)$ converges uniformly to $\eta_0^v(x)$. Hence equations (8.3) are true.

As a corollary of Lemmas 8.1 and 7.4 we have the following lemma.

LEMMA 8.2. *If $N_{ir}(x)$ are continuous functions which converge uniformly to $N_{i0}(x)$ on x^1x^2, and if $|g(x)|^2$ is integrable, then*

$$\lim \int_M N_{ir}(\dot{\eta}_r^i - \dot{\pi}_r^i)dx = \int_M N_{i0}(\dot{\eta}_0^i - \dot{\pi}_0^i)dx$$

for every measurable subset M of x^1x^2, and

$$\lim \int_M g(x)(\dot{\eta}_r^i - \dot{\pi}_r^i)dx = \int_M g(x)(\dot{\eta}_0^i - \dot{\pi}_0^i)dx$$

for every measurable subset M of x^1x^2 on which $\dot{y}_r^i$ converges uniformly.

LEMMA 8.3. *If $\phi(x, y, p)$ is any function of class C' near C_0, then*

$$\lim k_r^{-1}[\phi(x, y_r, p_r) - \phi(x, y_0, \dot{y}_0)] = \phi_{y^i}\eta_0^i + \phi_{p^i}\pi_0^i$$

uniformly on x^1x^2.

This follows directly from Taylor's theorem and Lemma 8.1.

If we replace, in Lemma 8.3, ϕ by ψ^ρ and then by ϕ^β and use equations (8.1) we immediately deduce the following lemma.

LEMMA 8.4. *The functions $\eta_0^i(x)$ satisfy with the auxiliary functions $\pi_0^i(x)$ the following equations:*

$$\psi_{y^i}^\rho\eta_0^i + \psi_{p^i}^\rho\pi_0^i = 0, \qquad \phi_{y^i}^\beta\eta_0^i + \phi_{p^i}^\beta\pi_0^i = 0.$$

9. **First order terms.** Let $H(x, y, p)$ be a function of class C' near C_0, and define

$$H(C, M) = \int_M H(x, y, \dot{y})dx,$$

$$H^*(C, M) = \int_M [H(x, y, P) + (\dot{y}^i - P^i)H_{p^i}(x, y, P)]dx,$$

$$E_H^*(C, M) = \int_M E_H(x, y, P, \dot{y})dx,$$

$$H_1(\eta, M) = \int_M (H_{y^i}\eta^i + H_{p^i}\dot{\eta}^i)dx.$$

It is clear that

$$H(C, M) = H^*(C, M) + E_H^*(C, M). \tag{9.1}$$

LEMMA 9.1. *If $H(x, y, p)$ is of class C' near C_0, then*

$$\lim k_r^{-1}[H^*(C_r, M) - H^*(C_0, M)] = H_1(\eta_0, M).$$

This follows at once from Lemmas 8.3 and 8.2.

Let us define

$$J_1(\eta) = \int_{x^1}^{x^2} (F_{y^i}\eta^i + F_{p^i}\dot{\eta}^i)dx.$$

By virtue of equation (2.3) and the fact that $\eta_r^i(x^s) = 0$, we have the following lemma.

LEMMA 9.2. *For each* $r = 0, 1, \cdots$, $J_1(\eta_r) = F_{p^i}\eta_r^i\big|_{x^1}^{x^2} = 0$.

LEMMA 9.3. *We have that*

$$\lim k_r^{-1}[J(C_r) - J(C_0)] = \lim k_r^{-1}[J^*(C_r) - J^*(C_0)] = \lim k_r^{-1}E^*(C_r) = 0.$$

By Lemmas 9.1 and 9.2 we have that

$$\lim k_r^{-1}[J^*(C_r) - J^*(C_0)] = J_1(\eta_0) = 0.$$

From equation (6.7) we conclude that

$$(9.2) \qquad 0 \geqq \limsup k_r^{-1}[J(C_r) - J(C_0)] = \limsup k_r^{-1}E^*(C_r).$$

However, $E^*(C_r) \geqq 0$ if r is so large that (x, y_r, p_r) is near enough to C_0 for (2.6) to hold, since both (x, y_r, p_r) and $(x, y_r, \dot{y}_r)$ are in $\mathcal{D}$ and $\phi^\beta(x, y_r, p_r) = 0$ if β is in $\Delta(x)$. Hence the right-hand side of (9.2) cannot be negative and so the lemma is true.

LEMMA 9.4. *If* $H(x, y, p)$ *is of class* C' *near* C_0 *and is* E^**-dominated by* F *near* C_0 *on* $\mathcal{D}$, *then*

$$\lim k_r^{-1}[H(C_r, M) - H(C_0, M)] = H_1(\eta_0, M).$$

If r is large enough, we may integrate the inequality (6.4) to see that $|E_H^*(C_r, M)| \leqq c^{-1}E^*(C_r)$ and so we see from Lemma 9.3 that

$$\lim k_r^{-1}E_H^*(C_r, M) = 0.$$

Lemma 9.4 is now an immediate consequence of Lemma 9.1 and equation (9.1).

10. Admissibility of the variation η_0^i. We have seen in Lemma 7.3 that the functions η_0^i are absolutely continuous, have integrable square derivatives, and satisfy the end conditions (2.8). We complete the proof of the admissibility of η_0^i in the following lemma.

LEMMA 10.1. *The variation* η_0^i *satisfies* (2.9) *for almost all* x *in* $B(\beta)$, (2.10) *for almost all* x *in* $A(\beta)$, *and* (2.11) *for almost all* x *on* x^1x^2.

It follows from Lemma 5.5 that the functions ψ^ρ and Φ^β satisfy the conditions imposed on H in Lemma 9.4. We thus infer that

$$0 = \lim k_r^{-1} \int_M [\psi^\rho(x, y_r, \dot{y}_r) - \psi^\rho(x, y_0, \dot{y}_0)]dx = \int_M (\psi^\rho_{y^i}\eta_0^i + \psi^\rho_{p^i}\dot{\eta}_0^i)dx$$

for every measurable subset M of x^1x^2. Hence (2.11) is satisfied for almost all x on x^1x^2. We also have that

$$(10.1)\quad \begin{aligned} 0 &\leqq \lim k_r^{-1} \int_M \phi^\beta(x, y_r, \dot{y}_r)dx = \int_M (\bar{\phi}^\beta_{y^i}\eta_0^i + \bar{\phi}^\beta_{p^i}\dot{\eta}_0^i)dx, \\ 0 &\leqq \int_M (\phi^\beta_{y^i}\eta_0^i + \phi^\beta_{p^i}\dot{\eta}_0^i)dx \end{aligned}$$

for every measurable subset M of $A(\beta)$. Hence (2.10) is satisfied for almost all x in $A(\beta)$. If M is a closed subset of $B(\beta)$, there is a number $\epsilon_\beta(M)$ such that $\lambda^\beta(x) \leqq -\epsilon_\beta(M) < 0$ on M. Hence it follows from the definition of k_r that

$$\begin{aligned} 0 \leqq k_r^{-1} \int_M \bar{\phi}^\beta(x, y_r, \dot{y}_r)dx &\leqq - k_r^{-1}\epsilon_\beta^{-1}(M) \int_{x^1}^{x^2} \lambda^\beta(x)\bar{\phi}^\beta(x, x_r, \dot{y}_r)dx \\ &\leqq k_r\epsilon_\beta^{-1}(M). \end{aligned}$$

Hence the inequality (10.1) is an equality for every closed subset M of $B(\beta)$. It follows that (2.9) is satisfied for almost all x in $B(\beta)$.

11. **Second order terms.** The second variation of $J^*(C)$ along C_0 is

$$J_2^*(\eta) = \int_{x^1}^{x^2} [2\omega(x, \eta, \pi_0) + 2(\dot{\eta}^i - \pi_0^i)\omega_{\pi^i}(x, \eta, \pi_0)]dx$$

in which 2ω is defined in (2.12). It is easy to see that

$$J_2^*(\eta) = J_2(\eta) - \int_{x^1}^{x^2} F_{p^ip^v}(\dot{\eta}^i - \pi_0^i)(\dot{\eta}^v - \pi_0^v)dx.$$

LEMMA 11.1. *We have that* $\lim k_r^{-2}[J^*(C_r) - J^*(C_0)] = (1/2)J_2^*(\eta_0)$.

This follows at once from Taylor's theorem and Lemmas 9.2 and 8.2.

LEMMA 11.2. *If $H(x, y, p)$ is a function of the form*

$$H = \theta[\psi^\rho(x, y, p)\psi^\rho(x, y, p) + \phi_0^\beta(x, y, p)\phi_0^\beta(x, y, p)],$$

in which θ is constant and $\phi_0^\beta = \lambda^\beta(x)\phi^\beta(x, y, p)$ (β not summed), then

$$\lim k_r^{-2} \int_M E_H(x, y_r, p_r, \dot{y}_r)dx = 0$$

for every measurable subset M of x^1x^2 on which $\dot{y}_r^i$ converges uniformly.

Since equations (8.1) hold, the lemma is equivalent to proving that

$$\lim k_r^{-2} \int_M \phi_0^\beta(x, y_r, \dot{y}_r)\phi_0^\beta(x, y_r, \dot{y}_r)dx = 0$$

for each β. Since $\dot{y}_r^i$ converges uniformly to $\dot{y}_0^i$ on M and (2.4) holds, there exists for each $\epsilon>0$ an index $R(\epsilon)$ such that

$$0 \geqq \phi_0^\beta(x, y_r, \dot{y}_r)[1 + \phi^{\beta^2}(x, y_r, \dot{y}_r)]^{1/2} \geqq -\epsilon$$

for each β if $r>R(\epsilon)$. By the definition of k_r, we have that if $r>R(\epsilon)$,

$$0 \leqq k_r^{-2} \int_M \phi_0^\beta(x, y_r, \dot{y}_r)\phi_0^\beta(x, y_r, \dot{y}_r)dx \leqq -k_r^{-2}\epsilon \int_M \lambda^\beta(x)\bar{\phi}^\beta(x, y_r, \dot{y}_r)dx \leqq \epsilon$$

for each β. Hence the lemma is true.

LEMMA 11.3. *We have that*

$$\liminf k_r^{-2}E^*(C_r) \geqq \frac{1}{2}\int_{x^1}^{x^2} F_{p^ip^v}(\dot{\eta}_0^i - \pi_0^i)(\dot{\eta}_0^v - \pi_0^v)dx.$$

In order to prove this result let M be a measurable subset of x^1x^2 on which $\dot{y}_r^i$ converges uniformly to $\dot{y}_0^i$. With the help of Lemma 11.2 and Corollary 3 to Theorem 4.1 the proof of the relation

$$\liminf k_r^{-2} \int_M E_F(x, y_r, p_r, \dot{y}_r, \lambda)dx \geqq \frac{1}{2}\int_M F_{p^ip^v}(\dot{\eta}_0^i - \pi_0^i)(\dot{\eta}_0^v - \pi_0^v)dx$$

can be made by the method of Hestenes [5, Lemma 10.1]. Since $\lambda^\beta(x)\phi^\beta(x, y_r, \dot{y}_r) \leqq 0$ and since (2.6) holds, we thus find that

$$\begin{aligned}\liminf k_r^{-2}E^*(C_r) &\geqq \liminf k_r^{-2} \int_M [E_F(x, y_r, p_r, \dot{y}_r, \lambda) - \lambda^\beta(x)\phi^\beta(x, y_r, \dot{y}_r)]dx \\ &\geqq \liminf k_r^{-2} \int_M E_F(x, y_r, p_r, \dot{y}_r, \lambda)dx \\ &\geqq \frac{1}{2}\int_M F_{p^ip^v}(\dot{\eta}_0^i - \pi_0^v)(\dot{\eta}_0^v - \pi_0^v)dx.\end{aligned}$$

Since $\dot{y}_r^i$ converges almost uniformly on x^1x^2, it follows from our choice of M and the integrability of $|\dot{\eta}_0|^2$ that this last inequality also holds when M is replaced by the whole interval x^1x^2. Hence the lemma is true.

12. **Completion of the proof of Theorem** 2.1. By virtue of the definition of C_r and equation (6.7), we have that

$$(12.1) \qquad 0 \geqq k_r^{-2}[J(C_r) - J(C_0)] = k_r^{-2}[J^*(C_r) - J^*(C_0) + E^*(C_r)].$$

By Lemmas 11.1, 11.3 and equation above Lemma 11.1, we have that

$0 \geqq J_2(\eta_0)$. Since η_0^i is an admissible variation by Lemmas 7.3 and 10.1, it follows from the hypothesis of Theorem 2.1 that $\eta_0^i(x)=0$. It then follows from (12.1) and the non-negativeness of $E^*(C_r)$ for sufficiently large r that $\lim k_r^{-2}E^*(C_r)=0$. By Lemma 5.5 there exists a positive number b^*, which we may assume to be less than one, such that $E_L(p_r, \dot{y}_r) \geqq b^* E_L(p_r - \dot{y}_0, \dot{y}_r - \dot{y}_0)$ for r sufficiently large. It then follows from (2.6) that for sufficiently large r,

$$k_r^{-2}E^*(C_r) \geqq bb^*k_r^{-2}\int_{x^1}^{x^2} [E_L(p_r - \dot{y}_0, \dot{y}_r - \dot{y}_0) - \lambda^\beta(x)\Phi^\beta(x, y_r, \dot{y}_r)]dx$$

$$\geqq bb^*k_r^{-2}\int_{x^1}^{x^2} \left[L(\dot{y}_r - \dot{y}_0) - \frac{1 + k_r^2\dot{\eta}_r^i\pi_r^i}{L(p_r - \dot{y}_0)} - \lambda^\beta(x)\Phi^\beta(x, y_r, \dot{y}_r)\right]dx$$

$$\geqq bb^*k_r^{-2}\int_{x^1}^{x^2} \left[L(\dot{y}_r - \dot{y}_0) - 1 - \lambda^\beta(x)\Phi^\beta(x, y_r, \dot{y}_r) - \frac{k_r^2\dot{\eta}_r^i\pi_r^i}{L(p_r - \dot{y}_0)}\right]dx.$$

By Lemmas 7.4 and 8.1 and the definition of k_r we find that

$$\liminf k_r^{-2}E^*(C_r) \geqq bb^* > 0$$

since $\eta_0^i \equiv 0$, and this is a contradiction from which we infer the truth of Theorem 2.1.

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